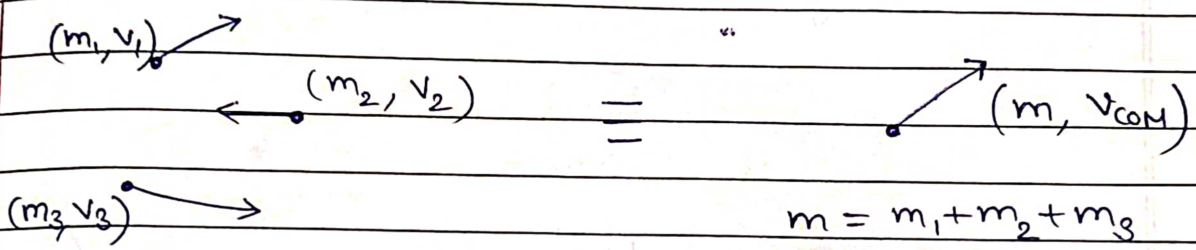
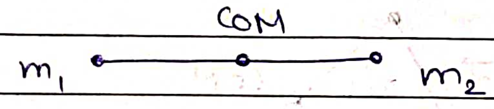
Centre of Mass

To convert multi particle system into single particle system.

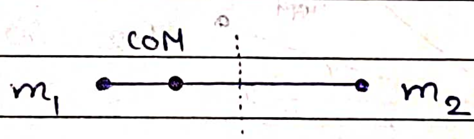


$m = m_1 + m_2 + m_3$

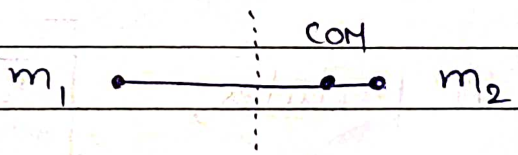
If $m_1 = m_2$,



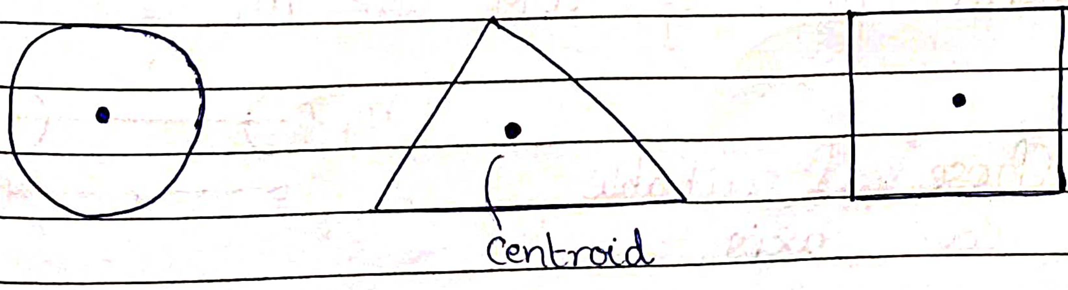
If $m_1 > m_2$,



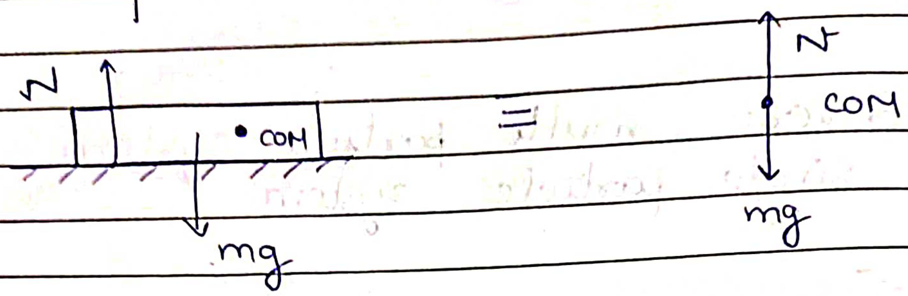
If $m_1 < m_2$,



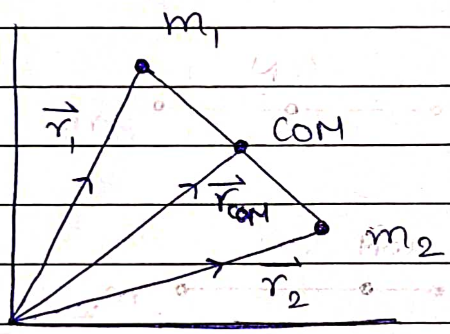
If body symmetric $\Rightarrow$ COM at centre.



For using laws of Motion, transfer all forces to COM.



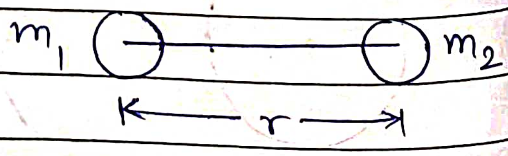
COM of 2 particle system



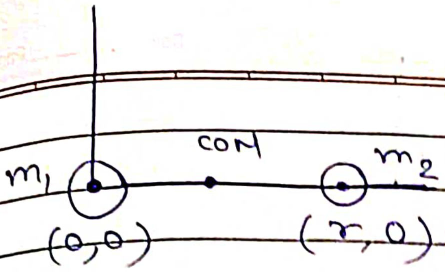
$$\boxed{\vec{r}_{COM} = \left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right)}$$

For 'n' particles, $\boxed{\vec{r}_{COM} = \left(\frac{\sum m_i \vec{r}_i}{\sum m_i}\right)}$

Q) Find posn. of COM wrt m_1.



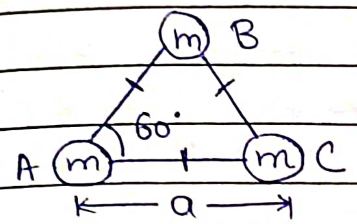
A) Chose a suitable coor. axis.



$$x_{COM} = \left(\frac{m_1(0) + m_2(r)}{m_1 + m_2}\right)$$

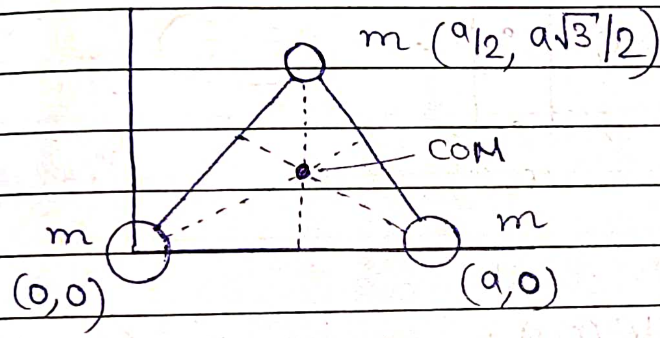
$$\Rightarrow \boxed{x_{COM} = \left(\frac{m_2 r}{m_1+m_2}\right)}$$

Q)



Find posn. of COM wrt. A.

A)

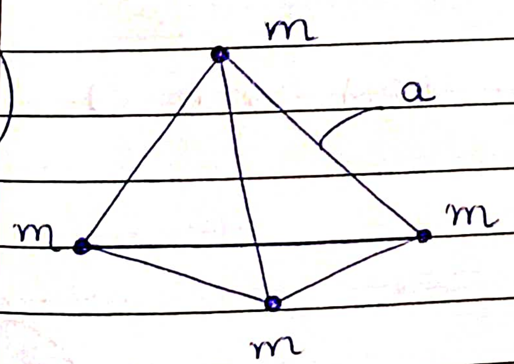


$$x_{COM} = m\left(\frac{0 + a/2 + a}{3m}\right)$$

$$y_{COM} = m\left(\frac{0+0+a\sqrt{3}/2}{3m}\right)$$

$$\Rightarrow \boxed{COM = \left(a/2,\ a/2\sqrt{3}\right)}$$

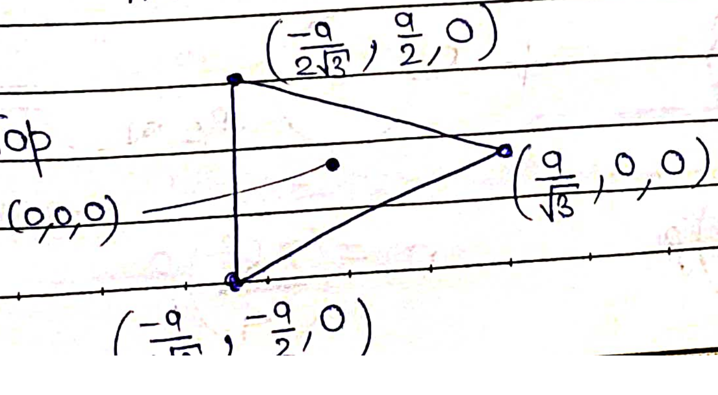
Q) ☆



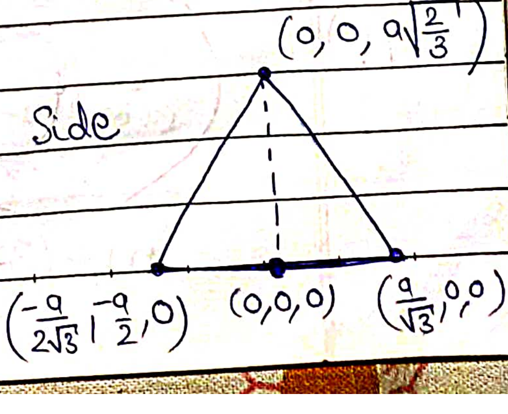
Given regular tetrahedron, find COM.

A)

Top



Side



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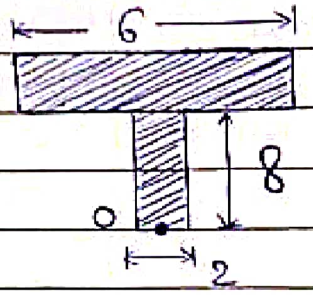
$$\boxed{COM \equiv \left(0,\ 0,\ \frac{a\sqrt{2}}{4\sqrt{3}}\right)}$$

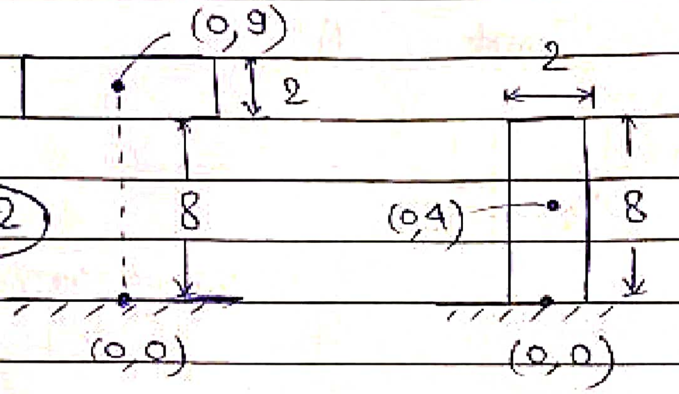
Q)



Uniformly distributed mass. find posn. of COM wrt. O

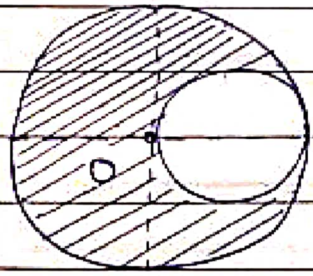
A)

Let $\rho = 1$



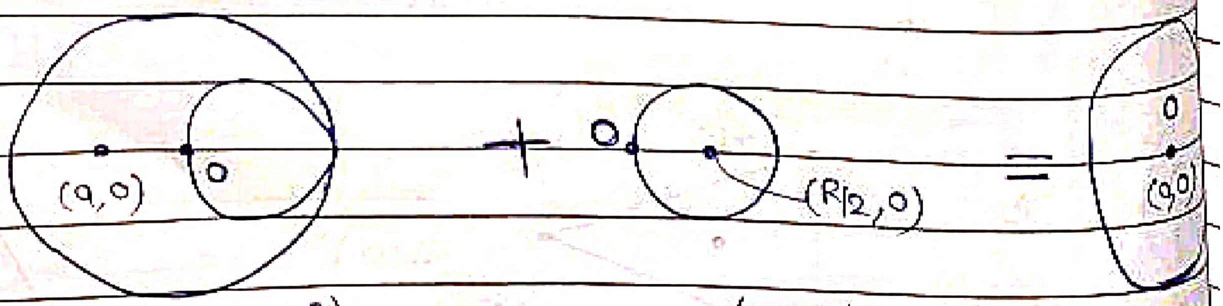
$$COM = \frac{12(0,9)+16(0,4)}{12+16} \Rightarrow \boxed{COM = \left(0,\ \frac{43}{7}\right)}$$

☆ Q)



Find COM wrt O

A)



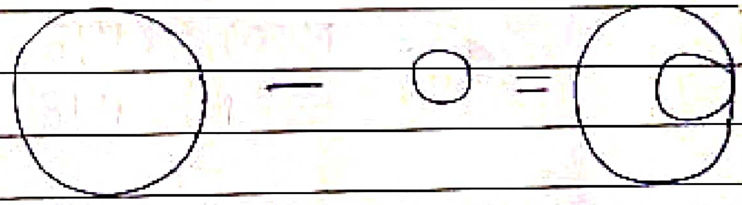
$$m = \left(\frac{3\pi R^2}{4}\right)\rho \qquad m = \left(\frac{\pi R^2}{4}\right)\rho$$

$$\langle 0,0\rangle = \frac{\left(\frac{3\pi R^2}{4}\right)\rho\langle a,0\rangle + \left(\frac{\pi R^2}{4}\right)\rho\left\langle \frac{R}{2},0\right\rangle}{(\pi R^2)\rho}$$

$$\Rightarrow \boxed{a = (-R/6)} \Rightarrow \boxed{COM \equiv \left(\frac{-R}{6}, 0\right)}$$

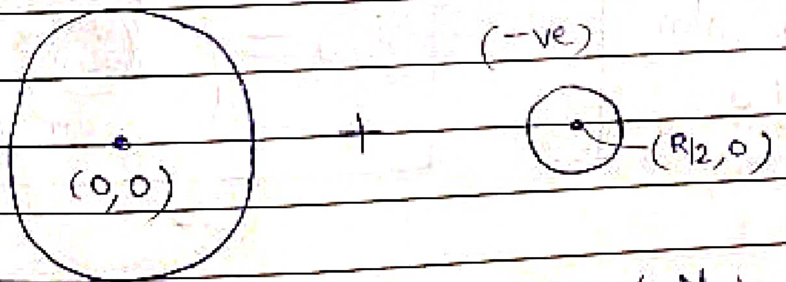
Concept of (−ve) Mass

Eg: In last Q,



$m = M$ $\quad m = \frac{M}{4}$ $\quad m = \frac{3M}{4}$

Since small disc removed,
we take mass of small disc (−ve)

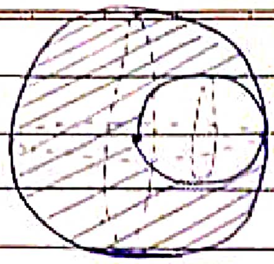


$m = M$ $\quad m = \left(\frac{-M}{4}\right)$

$$x_{COM} = \frac{M\langle 0,0\rangle - \frac{M}{4}\left\langle \frac{R}{2},0\right\rangle}{M - M/4}$$

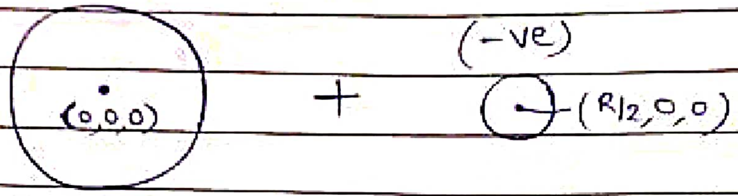
$$\Rightarrow \boxed{x_{COM} = \frac{-R}{6}}$$

Date:

Q) 

Solid spheres given

Find COM

A) 

$m = M$ $\qquad$ $m = (-M/8)$

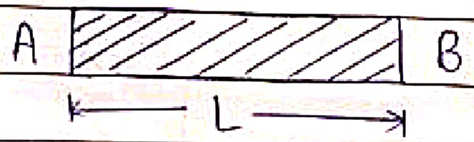
$$x_{COM} = \frac{M(0) - (M/8)(R/2)}{M - M/8} \Rightarrow \boxed{x_{COM} = \left(\frac{-R}{14}\right)}$$

COM of Non Uniform Distr. of Mass.

(x coor of COM of element)

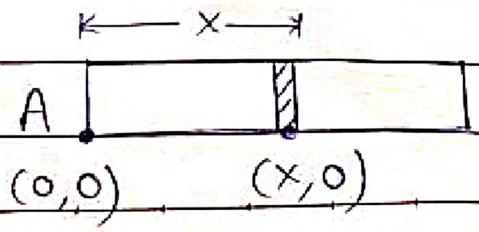
$$\boxed{x_{COM} = \left(\frac{\int x_{(element)}\, dm}{\int dm}\right)} \qquad \boxed{y_{COM} = \left(\frac{\int y_{(element)}\, dm}{\int dm}\right)}$$

(y coor of COM of element)

Q) 

$\lambda = 2x$ where x measured from A

Find COM.

A) 

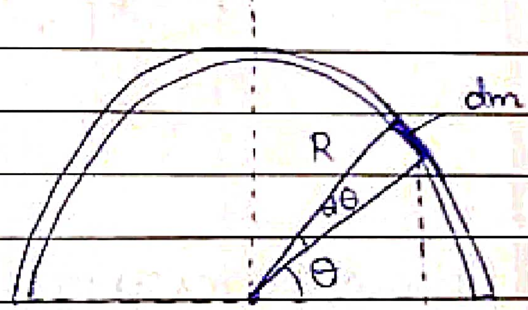
$\lambda = 2x \Rightarrow dm = 2x\, dx$

$$x_{COM} = \frac{\int x\,dm}{\int dm} = \frac{\int 2x^2\,dx}{\int 2x\,dx} = \left(\frac{x^3/3}{x^2/2}\right)_0^L$$

$$\Rightarrow \boxed{x_{COM} = \left(\frac{2L}{3}\right)}$$

COM of Diff. Shapes

1) Semi ⊙ Ring :



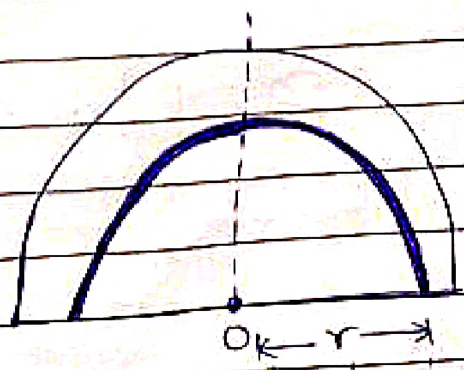
By Symmetry $x_{COM} = 0.$

$$\rho = \frac{M}{\pi R} = \frac{dm}{R\,d\theta} \Rightarrow dm = \left(\frac{M}{\pi}\right) d\theta$$

Now, $y_{COM} = \left(\frac{1}{M}\right)\left(\int R\, s_\theta\, dm\right) = \left(\frac{1}{M}\right)\left(\int_0^\pi \frac{MR}{\pi} s_\theta\, d\theta\right)$

$$\Rightarrow \boxed{y_{COM} = \left(\frac{2R}{\pi}\right)}$$

2) Semi ⊙ Disc :



By symmetry $x_{COM} = 0.$

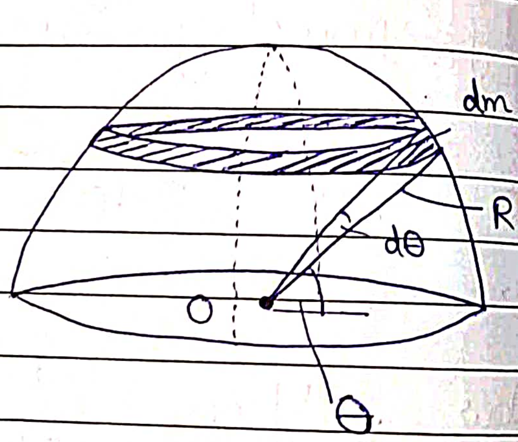
$$\rho = \left(\frac{2M}{\pi R^2}\right) = \left(\frac{dm}{\pi r\, dr}\right) \Rightarrow \left(dm = \left(\frac{2M}{R^2}\right) r\, dr\right)$$

Now, $\quad y_{COM} = \left(\frac{1}{M}\right)\left(\int \left(\frac{2r}{\pi}\right) dm\right)$

$$= \left(\frac{1}{M}\right)\left(\int_0^R \frac{2r}{\pi} \cdot \frac{2M}{R^2}\, r\, dr\right)$$

$$= \left(\frac{4}{\pi R^2}\right) \int_0^R r^2\, dr \Rightarrow \boxed{y_{COM} = \left(\frac{4R}{3\pi}\right)}$$

3) Hollow Hemisphere :



By symmetry $x_{COM} = 0$.

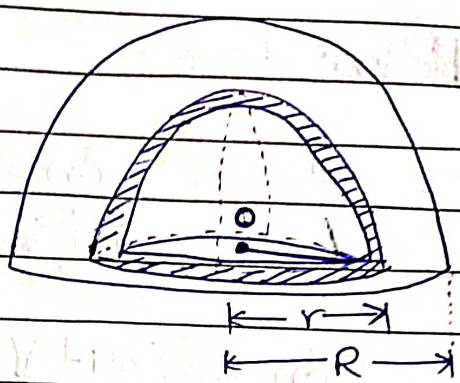
$$\rho = \left(\frac{M}{2\pi R^2}\right) = \left(\frac{dm}{2\pi R \cos\theta \cdot R\, d\theta}\right)$$

$$\Rightarrow \quad dm = M \cos\theta\, d\theta$$

Now, $\quad y_{COM} = \left(\frac{1}{M}\right)\left(\int R \sin\theta\, dm\right) = \left(\frac{R}{2}\right)\int_0^{\pi/2} \sin 2\theta\, d\theta$

$$\Rightarrow \quad \boxed{y_{COM} = \left(\frac{R}{2}\right)}$$

4) Solid Hemisphere :



By symmetry $x_{COM} = 0$.

$$\rho = \frac{M}{\left(\frac{2}{3}\pi R^3\right)} = \frac{dm}{2\pi r^2 \, dr}$$

$$\Rightarrow \quad \boxed{dm = \left(\frac{3M}{R^3}\right) r^2 \, dr}$$

Now, $$y_{COM} = \left(\frac{1}{M}\right)\left(\int \frac{r}{2} \, dm\right) = \left(\frac{3}{2R^3}\right)\int_0^R r^3 \, dr$$

$$\Rightarrow \quad \boxed{y_{COM} = \left(\frac{3R}{8}\right)}$$

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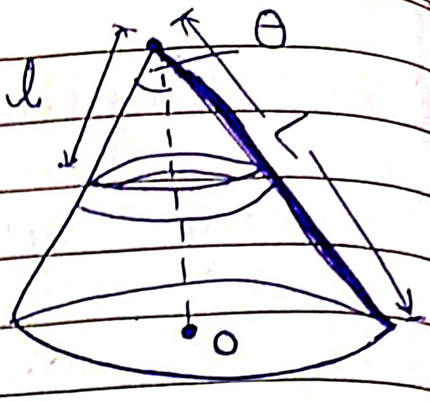
5) Hollow Cone :



$$\rho = \frac{M}{\pi L^2 s_\theta} = \frac{dm}{2\pi l s_\theta \, dl}$$

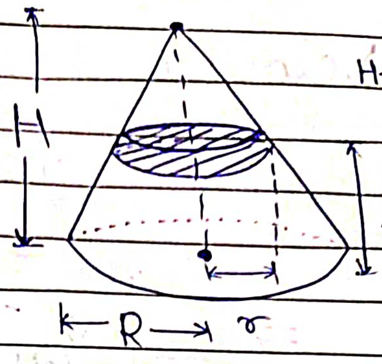
$$\Rightarrow \boxed{dm = \left(\frac{2M}{L^2}\right) l \, dl}$$

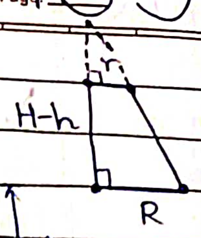
Now, $y_{COM} = \left(\frac{1}{M}\right)\left(\int (L-l) s_\theta \, dm\right)$

$$= \left(\frac{1}{M}\right)\left(\int_0^L (L-l)\left(\frac{2M s_\theta}{L^2}\right) l \, dl\right) = \left(\frac{L s_\theta}{3}\right) = \frac{H}{3}$$

$$\Rightarrow \boxed{y_{COM} = \left(\frac{H}{3}\right)}$$

5) Solid Cone :





$$\rho = \frac{M}{\frac{1}{3}\pi R^2 H} = \frac{dm}{\pi r^2\, dh}$$

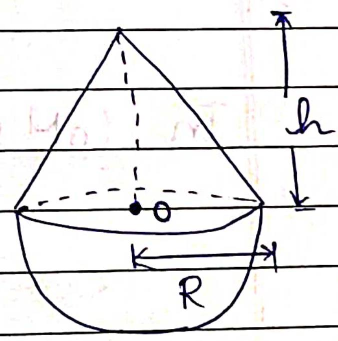
$$\Rightarrow \boxed{dm = \left(\frac{3M}{H^3}\right)(H-h)^2\, dh}$$

$$r = \left(\frac{R}{H}\right)(H-h)$$

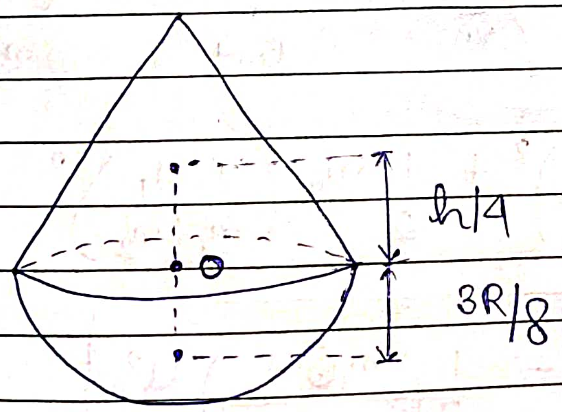
Now, $y_{COM} = \left(\frac{1}{M}\right)\left(\int h\, dh\right) = \left(\frac{1}{M}\right)\left(\int_0^H \left(\frac{3M}{H^3}\right)(H-h)^2 h\, dh\right)$

$$= \left(\frac{3}{H^3}\right)\left(\int_0^H h^2(H-h)\, dh\right) \Rightarrow \boxed{y_{COM} = \left(\frac{H}{4}\right)}$$

Q) COM of given solid at O. Find h/R.



A)



$$M_{cone} = \frac{1}{3}\pi R^2 h$$

$$M_{hemi} = \frac{2}{3}\pi R^3$$

$$y_{COM} = \left(\frac{1}{3}\right)\pi R^2 h \cdot \frac{h}{4} - \left(\frac{2}{3}\right)\pi R^3 \cdot \frac{3R}{8} = 0$$

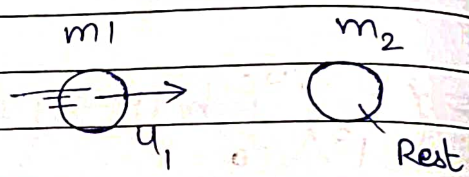
$$\Rightarrow \boxed{h/R = \sqrt{3}}$$

Velocity of CoM –

$$\vec{r}_{CoM} = \left(\frac{\sum m_i r_i}{\sum m_i}\right)$$

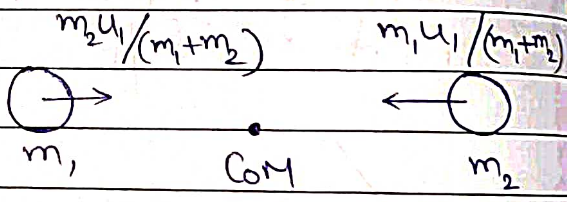
$$\Rightarrow \boxed{\vec{v}_{CoM} = \left(\frac{\sum m_i \vec{v}_i}{\sum m_i}\right)}$$

Q) Find KE of system wrt CoM.



A) $\vec{v}_{CoM} = \left(\frac{m_1 u_1}{m_1+m_2}\right)$ in dirx^n of m_1's motion.

In CoM's frame,

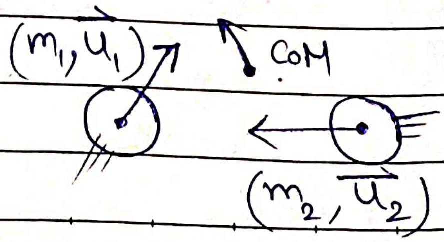


$$KE_{system} = \frac{1}{2} m_1 \left(\frac{m_2 u_1}{m_1+m_2}\right)^2 + \frac{1}{2} m_2 \left(\frac{m_1 u_1}{m_1+m_2}\right)^2$$

$$\Rightarrow \boxed{KE_{system} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right) u^2}$$

☆ While in CoM's frame,

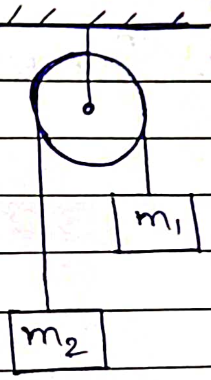
$$\boxed{K_{system \text{ wrt CoM}} = \frac{1}{2}\underbrace{\left(\cancel{\frac{1}{m_1}+\frac{1}{m_2}}\right)}_{\left(\frac{m_1 m_2}{m_1+m_2}\right)}|\vec{u}_1-\vec{u}_2|^2}$$



Acceleration of CoM –

$$\boxed{\vec{a}_{COM} = \left(\frac{\sum m_i \vec{a}_i}{\sum m_i}\right)}$$

Q) Find a_{COM}, given $m_1 > m_2$.



A) m_1 falls down and m_2 goes up, both with acc. a.

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

(m_2, a) ↑ ↓ (m_1, a)

$$a_{COM} = \left(\frac{m_1 a - m_2 a}{m_1 + m_2}\right) \Rightarrow \boxed{a_{COM} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g}$$

Centre of Gravity

$$\boxed{\vec{r}_{CoG} = \left(\frac{\sum W_i \vec{r}_i}{\sum W_i}\right)}$$

W_i = Weight of ith particle.

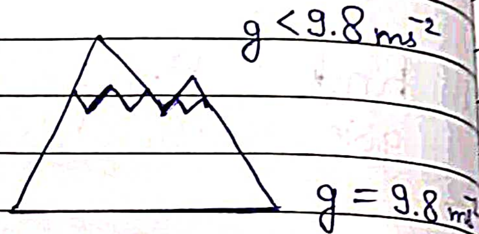
Date:

Generally, $\boxed{\vec{r}_{CoM} = \vec{r}_{CoG}}$

But for massive obj., s.t. with great height value of 'g' changes with height,

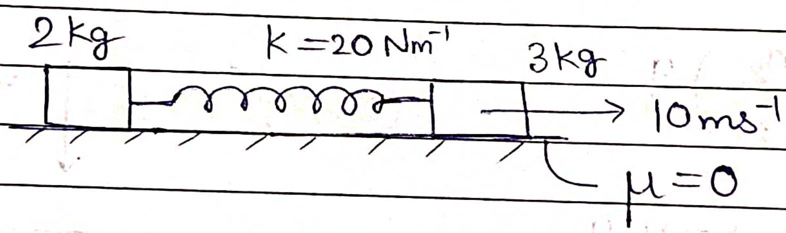
$\boxed{CoG \not\equiv CoM}$

Eg: Mt. Everest



☆

Q) Find max. ~~compression~~ extension in spring.



A) ☆ At max. extension or compression both obj. same vel.

We see system in CoM's frame.

System: $K_1 + U_1 = K_2 + U_2$

$$\Rightarrow \frac{1}{2}\left(\cancel{\frac{1}{m_1}+\frac{1}{m_2}}\right)(u_1-u_2)^2 = \frac{1}{2}Kx_{max}^2.$$

$$\left(\frac{m_1 m_2}{m_1+m_2}\right)$$

$$\Rightarrow \left(\frac{1}{2}\right)\left(\frac{2\cdot 3}{(2+3)}\right)(10-0)^2 = \frac{1}{2}\cdot 20\cdot x_{max}^2 \Rightarrow x_{max} = \sqrt{6}$$

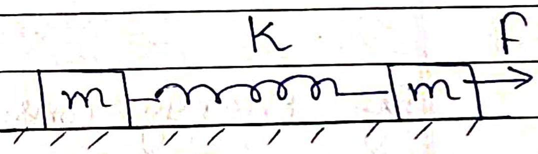
Another Method: Conserve momentum, as no F_{ext}.

$$3\cdot 10 + 2\cdot 0 = (2+3)\,v_f \Rightarrow v_f = 6$$

$$\Delta K + \Delta U = 0 \Rightarrow \frac{1}{2}\cdot 3\cdot 6^2 + \frac{1}{2}\cdot 2\cdot 6^2 - \frac{1}{2}\cdot 3\cdot 10^2 + \frac{1}{2}\cdot 20\cdot x^2 = 0$$

$$\Rightarrow x = \sqrt{6}$$

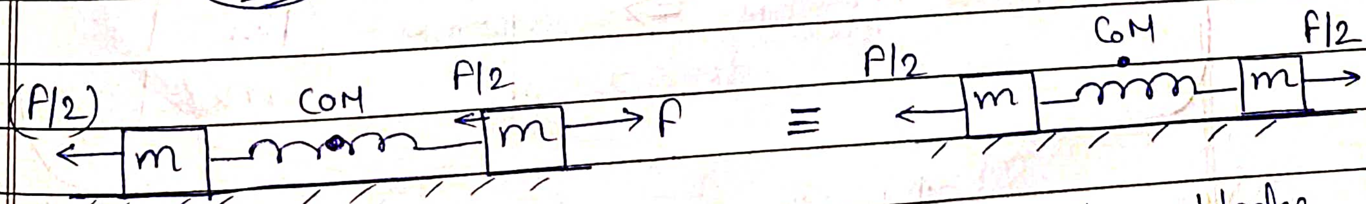
Q) Find max. extension.



A) We see system wrt. COM.

$$a_{COM} = \frac{(F/m)(m)}{(2m)} \Rightarrow a_{COM} = \left(\frac{F}{2m}\right) \text{ in } F\text{'s dirx}^n$$

Both ~~block~~ block experience pseudoforce.



In COM's frame, max. extension when blocks at rest.

$$W_F = \Delta U \Rightarrow \frac{Fx_1}{2} + \frac{Fx_2}{2} = \frac{1}{2}K(x_1+x_2)^2 \Rightarrow \boxed{x_{max} = \left(\frac{F}{K}\right)}$$

(x_1 – Ext. of m_1
x_2 – Ext. of m_2)

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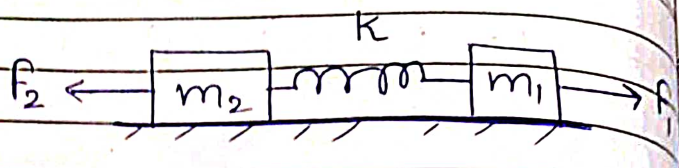
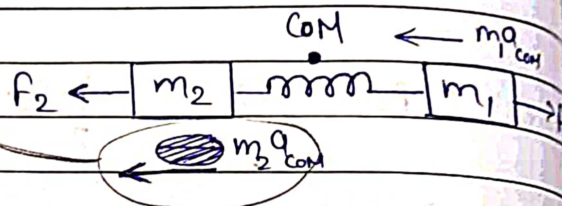
Q) find max. extension



A) $a_{CoM} = \dfrac{m_1(F_1/m_1) - m_2(F_2/m_1)}{m_1+m_2} = \left(\dfrac{F_1-F_2}{m_1+m_2}\right)$

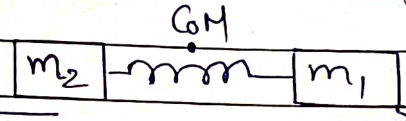
Let WLOG $F_1 > F_2$.

In CoM's frame,



(Net force on system towards F_1)

$\Rightarrow$



$\left(\dfrac{m_1F_2+m_2F_1}{m_1+m_2}\right)$ $\qquad$ $\left(\dfrac{m_1F_2+m_2F_1}{m_1+m_2}\right)$

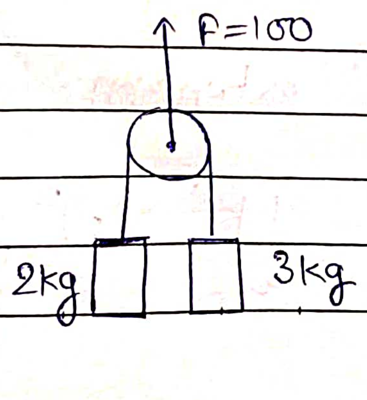
By Energy Consrv., $\left(\dfrac{m_1F_2+m_2F_1}{m_1+m_2}\right)(x_1) + \left(\dfrac{m_1F_2+m_2F_1}{m_1+m_2}\right)(x_2)$

$= \dfrac{1}{2}K(x_1+x_2)^2$

$\Rightarrow$ $\boxed{x_{max.} = \dfrac{2(m_1F_2+m_2F_1)}{K(m_1+m_2)}}$

☆ $F_{ext.} = m_{System}\, a_{CoM} \Rightarrow \boxed{a_{CoM} = \left(\dfrac{F_{ext.}}{m_{System}}\right)}$

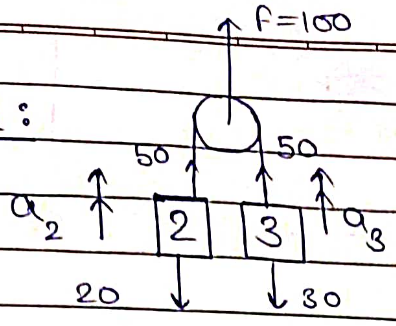
Q) find a_{CoM}



A) $a_{CoM} = \left(\dfrac{100-20-30}{2+3}\right)$

$\Rightarrow$ $\boxed{a_{CoM} = 10}$

Another Method :



$$a_3 = \left(\frac{50-30}{3}\right)$$

$$a_2 = \left(\frac{50-20}{2}\right)$$

$$a_{CoM} = \frac{3(20/3) + 2(30/2)}{3+2} \Rightarrow \boxed{a_{CoM} = 10}$$

If $\underline{F_{ext} = 0} \Rightarrow \underline{a_{CoM} = 0} \Rightarrow \underline{V_{CoM} = \text{Const}}$

$\boxed{V_{CoM} = 0}$

$\Rightarrow X_{CoM} = \text{Const.}$

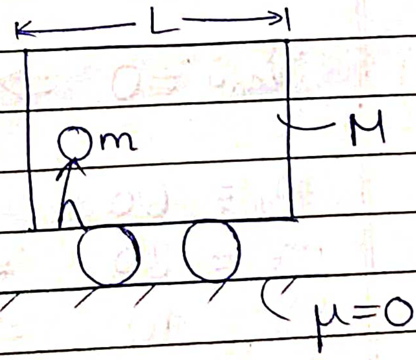
$\Rightarrow \Delta X_{CoM} = 0$

$\boxed{V_{CoM} \neq 0}$

$\Rightarrow X_{CoM} \neq \text{Const.}$

$\Rightarrow \Delta X_{CoM} \neq 0$

Q) If person moves from one end to another of wagon, find dist. moved by wagon. System initially at rest.



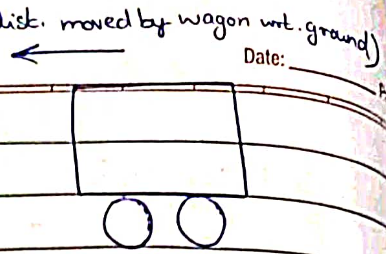
A) $V_{CoM} = 0$ (as $U_{CoM} = 0$ & $F_{ext} = 0$) $\Rightarrow \Delta X_{CoM} = 0 = \left(\frac{m\Delta X_1 + M\Delta X_2}{m+M}\right)$

$\Rightarrow \underbrace{m\Delta X_1}_{\text{Shift in CoM of m wrt ground}} + \underbrace{M\Delta X_2}_{\text{Shift in CoM of M wrt ground}} = 0 \Rightarrow \Delta X_2 = x$ (dist. moved by wagon)

L (dist by man wrt wagon)

$\Rightarrow \Delta X_1 = L - x$

Date:

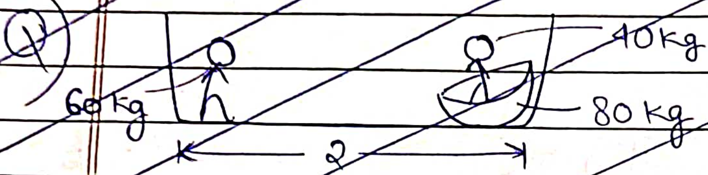


and $\Delta x_2 = x$

$\Rightarrow$ $M x + m(L-x) = 0$

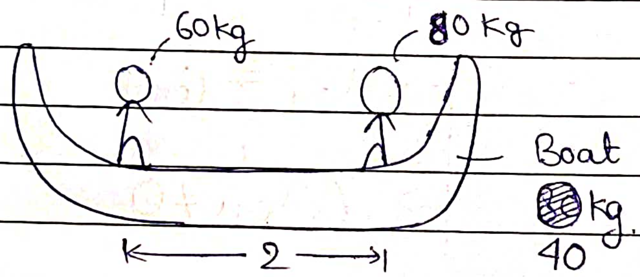
$\Rightarrow$ $\boxed{x = \left(\frac{mL}{M+m}\right)}$

~~Q)~~



~~If people exchange position, find dist. moved by boat. System at rest.~~

Q) ★



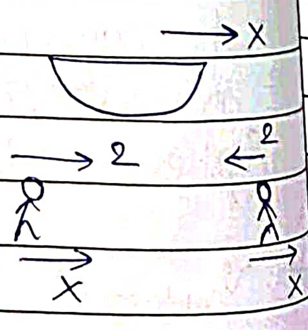
If people exchange post., find dist. moved by boat, if system initially at rest.

A) $V_{COM} = 0 \Rightarrow m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 = 0$

$m_1 = 60 \Rightarrow \Delta x_1 = 2 + x$

$m_2 = 80 \Rightarrow \Delta x_2 = x - 2$

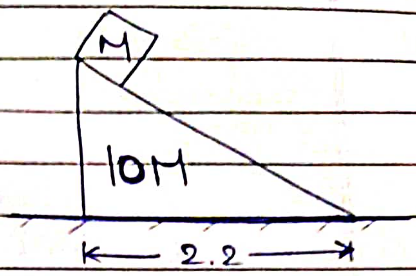
$m_3 = 40 \Rightarrow \Delta x_3 = x$



$\Rightarrow 60(2+x) + 80(x-2) + 40(x) = 0$

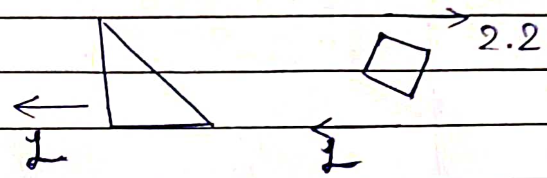
$\Rightarrow \boxed{x = 2/9}$

Q) find dist. moved by 10m mass when m mass reaches bottom of incline.



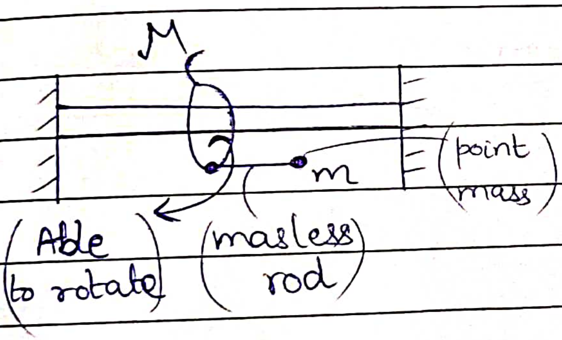
A) $\Delta X_{CoM} = 0 = 10M(L) + M(L - 2.2)$

$\Rightarrow \boxed{L = 0.2}$



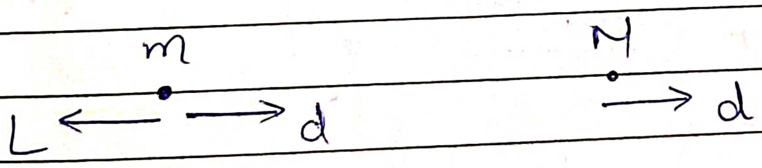
Fext only in Y dirxⁿ!

Q)



If length of rod is L, find dist. moved by mass M when rod becomes vertical.

A)



$\Delta X_{CoM} = 0$

$\Rightarrow m(d-L) + Md = 0$

$\Rightarrow \boxed{d = \left(\frac{mL}{M+m}\right)}$

Fext only in Y dirxⁿ!